

A Novel Phase Shift Acquired due to Virtual Forces

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Abstract

In the following we discuss a simple one dimensional scattering problem involving a strong short-range interaction between a heavy and a light particle. It allows to introduce concepts of "potential", virtual forces and "private" potentials, which explain the phase acquired by the heavy particle during the entanglement /scattering event.

I. Introduction

Locality has been introduced first into physics by Faraday and Maxwell. It implies that a particle is affected by the forces acting at its own location rather than by an "action at a distance". However, in quantum theory, one may need to consider not only the forces acting directly on the external particle, but also the forces acting between its disjoint supports. Thus, to preserve this all important principle one has to extend the notion to local measurements performed in the simply connected region in which the particle's trajectories can be embedded.

In an article by Stern, Aharonov and Imry [1], motivated by Furry and Ramsey's paper [2], locality has been applied to an "external" particle which is entangled with its environment. In this context locality entails that we further restrict ourselves to measurements of the external particle alone. The discussion in [1] was restricted to the special case where the interaction affects no change of the source, i.e; the "internal" degree of freedom on which the interaction depends.

In this paper we apply the principle of locality to an external, heavy, *strongly* interacting particle that *modifies* the source of the interaction. A case in point is the interaction of a heavy, strongly charged particle, with a neutral atom. The charged particle polarizes the atom, inducing an electric dipole, and this in turn changes the potential seen by the particle. After completing a closed trajectory the external particle may, at most, pick up a relative phase. It will turn out that the present work amounts to a far reaching generalization of the electric Aharonov Bohm[AB] effect [3].

In the electric AB effect an "external" electron is prepared in a wave packet ψ . The latter is split into two equal wave packets of width Δx each, which are then separated by a distance L , $L \gg \Delta x$, and brought to rest. Thus the electron is initially in the state $\Psi_{in} = \frac{1}{\sqrt{2}}[\psi(x) + \psi(x-L)] \equiv \frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$. At $t = t_0$, an infinite (in the y, z , directions) parallel-plate condenser is "opened" in the interval between the packets. See fig. 1. [4]. The distance between the plates is $d \ll L$. At a later time $t_1 > t_0$ the condenser plates are brought back together and the electric field generated by it disappears. The two wave packets are then brought back together and allowed to interfere. Note that throughout the entire experiment the electron remains in a force-free region, yet there is a non vanishing potential difference $V(t)$ between the regions to the right and left of the condenser, where the wave packets reside. It turns out that because of the presence of the electric field in the region which is inaccessible to the electron, i.e; between the condensers' plates, a relative phase is generated such that $\Psi(t \geq t_1) = \frac{1}{\sqrt{2}}[\psi_1 + e^{i\frac{e}{\hbar} \int_{t_0}^{t_1} V(t)dt} \psi_2]$, resulting in a shift of the interference pattern.

In the above original electric AB effect, the external electron is viewed as a weakly charged test particle that does not change the distance between the condenser's plates and therefore does not modify the interaction [5]. This is not the case for the strongly charged particle considered in this paper. Here the relative phase picked up by the external particle after completing (in time T) a closed space-time circuit is

$$\varphi_{rel}(T) - \varphi_{rel}(0) = \int_0^L \int_0^T \bar{F} dx dt. \quad (1)$$

Note that \bar{F} appearing in the last expression is a *potential* force, measurable only if the external particle were shifted from its original location and placed at all possible intermediate points. This force, in turn, derives from a "private" potential, a novel concept which will play a key role in the following.

The above principle of locality does not conflict with the nonlocal aspects of a single quantum particle with a wave function made up of several distinct wave packets separated by the same distance L . These features are best described by modular momentum [6]. This momentum is represented by

a displacement operator $e^{\pm ipL}$ which maps, for the special case of two wave packets, the right wave packet onto the left and vice versa, thereby capturing the relative phase of the superposition. Modular momentum is also the key to the solution of the problem of which-way measurements, a subject of ongoing controversy, as will be shown in a forthcoming article.

Consider next a charged particle of mass M initially in the state $\Psi(0) = \frac{1}{\sqrt{2}}(\psi_1 + e^{i\varphi(0)}\psi_2)$, where $\psi_1(x)$, $\psi_2(x) = \psi_1(x - L)$, are two narrow (i.e., $\Delta x \ll L$) wave packets centered around $x = 0$ and $x = L$, respectively. The momentum modulo $\frac{\hbar}{L}$ is then given by $\overline{e^{ipL}}(0) = \frac{e^{i\varphi(0)}}{2}$. (The bar in this expression refers to the quantum average in the initial state $\Psi(0)$, which is the analog of the classical value.) With the Hamiltonian $H = \frac{p^2}{2M} + V(x)$, the time evolution of the modular momentum [7]

$$\frac{d}{dt}e^{ipL} = i[H, e^{ipL}] = i[V(x) - V(x + L)]e^{ipL} = \left(-i \int_x^{x+L} E dx\right) e^{ipL}. \quad (2)$$

is manifestly nonlocal. For a very heavy mass M we can approximate $H \approx V(x)$ allowing an exact solution of equation (2) above yielding [7]

$$e^{ipL}(t) = e^{i[V(x) - V(x+L)]t} e^{ipL}(0) = e^{-i\left(\int_x^{x+L} E dx\right)t} e^{ipL}(0). \quad (3)$$

Let $V(x) = V_0\theta(x - \frac{L}{2})$, where $\theta(x)$ is the step function. Note that in this case the particle is confined to a force free region. Nevertheless, the potential difference generates a phase shift between the two wave packets. Specifically,

$$\overline{e^{ipL}}(t) = \frac{e^{i[\varphi(0) - V_0 t]}}{2} = \frac{e^{i\varphi(t)}}{2}, \quad (4)$$

where $\varphi(t)$ is the relative phase of the state in the Schrödinger representation. In the electric AB effect the modified interference pattern can be viewed as due to a nonlocal exchange of modular momentum. This conforms with viewing the effect as due to the difference of the electric potentials between the two possible trajectories of the particle which moves exclusively in a force-free region. Alternatively, according to equations (1, 2) the phase is generated by integrating the local electric field over the interval between the wave packets.

Below, the new concepts are introduced using a one dimensional exactly solvable scattering problem.

II. The Scattering Problem

Consider the following one dimensional set-up. It consists of two particles: an "external", heavy particle, of mass M , and an "internal", light particle,

of mass $m \ll M$. The heavy particle, h , is initially in the superposition

$$\Psi_{h,in} = \frac{1}{\sqrt{2}}(\psi_1 + \psi_2), \quad (5)$$

where $\psi_1(x_h)$, $\psi_2(x_h) = \psi_1(x_h - L)$, are two similar, narrow ($\Delta x \ll L$), wave packets, separated by a distance L . The internal, light particle, l , is initially in a Gaussian wave packet $\phi_{in}(x_l, t)$ of width $W \gg L$, moving with velocity v_0 in the positive x -direction inside a long, narrow tunnel [8]. See fig. 2. The two particles interact strongly. This interaction is short-range and we approximate it by a δ function. $V(x_h, x_l) = \alpha\delta(x_l - x_h)$. For $\alpha \rightarrow \infty$ the light particle is totally reflected, and complete entanglement of the two particles obtains. We take M sufficiently large so that the recoil of the heavy particle is negligible and only momentum but no energy is exchanged. Ignoring then the constant kinetic energy of the heavy particle yields the Hamiltonian:

$$H = \frac{p_l^2}{2m} + \alpha\delta(x_l - x_h). \quad (6)$$

The state of the system at all times t is

$$\Psi(t) = \frac{1}{\sqrt{2}}[\psi_1\phi_1(t) + \psi_2\phi_2(t)], \quad (7)$$

where $\psi(\phi)$ refers to the heavy(light) particle respectively. Since we neglected the kinetic energy of the heavy particle, its two wave functions ψ_1 and ψ_2 are constant in time. Assuming no recoil of the heavy particle, the Hamiltonian (6) can be approximated by [9]

$$H = \frac{p_l^2}{2m} + \frac{1 + \sigma_3}{2}\alpha\delta(x_l) + \frac{1 - \sigma_3}{2}\alpha\delta(x_l - L), \quad (8)$$

where $|\sigma_3 = +1\rangle \equiv |\psi_1\rangle$, $|\sigma_3 = -1\rangle \equiv |\psi_2\rangle$. Within this approximation the equations for ϕ_1 and ϕ_2 separate, and the problem is exactly solvable for any α [10]. When $\alpha \rightarrow \infty$ the light particle is totally reflected from either δ functions.

The assumed large width W allows us to consider both wavepackets ϕ_1 and ϕ_2 as energy eigenstates with energy $\frac{p_0^2}{2m}$ and incident momentum $+p_0$. We denote by $\phi_{in}(x_l) \approx e^{ip_0x_l}$ and by $\phi_{out} \approx e^{-ip_0x_l}$ the initial and final states of the light particle. When $\alpha \rightarrow \infty$, the above incident state is completely reflected from either of the two δ functions [10]. Conservation of energy then decrees that it is reflected with momentum $-p_0$ and at most a change of phase. Specifically, the reflection coefficients from $\alpha\delta(x_l)$ and $\alpha\delta(x_l - L)$ are -1 and $-e^{2ip_0L}$ respectively [10]. For the actual wide ($W \gg L$) incident wave packet (see fig. 3) the average momentum is p_0 such that $\frac{p_0L}{\hbar} \gg 1$ and uncertainty $\Delta p_l \sim \frac{\hbar}{W} \ll \frac{\hbar}{L}$. The finite width W causes a negligible uncertainty in the relative phase $\Delta\varphi_{rel} = \frac{\Delta p_l 2L}{\hbar} \sim \frac{2L}{W} \ll 1$. Thus, for $t \gg T = \frac{W}{v_0}$, after the

reflection is over, the system is left in a new state $\Psi_f = \frac{1}{\sqrt{2}}(\psi_1 + e^{2ip_0L}\psi_2)\phi_{out}$ i.e; with a shift in the relative phase between the two wave packets ψ_1 and ψ_2 of the heavy particle. If for further simplicity we choose $2p_0L = \pi$, the complete wave function of the system at all times t is

$$\Psi(t) = \sqrt{\frac{T-t}{T}} \frac{\psi_1 + \psi_2}{\sqrt{2}} e^{ip_0x_l} + \sqrt{\frac{t}{T}} \frac{\psi_1 - \psi_2}{\sqrt{2}} e^{-ip_0x_l}. \quad (9)$$

Note that for $t = 0$ $\Psi(0) = \frac{\psi_1 + \psi_2}{\sqrt{2}} e^{ip_0x_l}$ reproduces the initial condition (5). In deriving equation (9) we assumed that the reflection of a single particle in the incident wave packet of width W from the two δ functions occurs via an instantaneous exchange of modular momentum. The specific time when this exchange occurs is uncertain, within $\Delta t = T$. Thus the entangled state survives for a time T during which the external particle's modular momentum is uncertain. The condition for having uncertain modular momentum, $|\overline{e^{ip_hL}}| < \frac{1}{2}$, is satisfied during that time interval T . Indeed, from equation (9), $\overline{e^{ip_hL}}(t) = \frac{T-t}{T}(\frac{1}{2}) + \frac{t}{T}(-\frac{1}{2})$. A fraction $\frac{t}{T}$ of the particles have already exchanged modular momentum, while the remaining $\frac{T-t}{T}$ still carry the "old" value $+\frac{1}{2}$. In general, with φ_1, φ_2 the initial and final relative phases, we have $|\overline{e^{ip_hL}}(t)| \leq \frac{T-t}{2T} |e^{i\varphi_1}| + \frac{t}{2T} |e^{i\varphi_2}| \leq \frac{1}{2}$. Note that the same inequality obtains from the exact solution (7). Then $\overline{e^{ip_hL}} = \frac{1}{2} < \phi_1(t) | \phi_2(t) > \leq \frac{1}{2}$. The orthogonality of the ϕ 's then introduces an uncertainty in the heavy particle's modular momentum. This connects the local picture, where we limit ourselves to measurements of the external particle alone, with the underlying entanglement.

The following paradox arises if we try to explain the change of the heavy particle's modular momentum, i.e; the acquired phase, while adhering to the principle of locality. Using equation (2) we conclude that there must be an average potential difference between $x = L$ and $x = 0$. Yet a simple calculation [11] reveals that as $\alpha \rightarrow \infty$ this potential difference vanishes. It appears as though modular momentum has unaccountably changed.

To get to the bottom of the mystery let us consider more carefully the heavy particle's modular momentum. From $\overline{e^{ip_hL}}(t) = \frac{1}{2} < \psi_1(x_h) | \psi_2(x_h + L) > < \phi_1(t) | \phi_2(t) > = \frac{1}{2} < \phi_1(t) | \phi_2(t) >$ it follows that [9, 12, 13]

$$\frac{d}{dt} \overline{e^{ip_hL}} = \frac{1}{2} [< \dot{\phi}_1(t) | \phi_2(t) > + < \phi_1(t) | \dot{\phi}_2(t) >] \quad (10)$$

$$= \frac{i}{2} < \phi_1(t) | \alpha \delta(x_l) - \alpha \delta(x_l - L) | \phi_2(t) > \quad (11)$$

$$= \frac{i}{2} < \phi_1(t) | \alpha \delta(x_l) | \phi_2(t) > = -\frac{v_0}{l} \left(\frac{1}{2} - \frac{e^{2ip_0L}}{2} \right), \quad (12)$$

where v_0 is the velocity of the internal particle. Note that since ϕ_1 is totally reflected at the origin, it does not reach $x = L$. Hence the con-

tribution of the reflection at $x = L$ vanishes. Thus only the first potential $\alpha\delta(x_l)$ in equation (11) contributes. Integrating (12) yields [13] $\overline{e^{ip_h L}}(t) = \frac{W-v_0 t}{W} \left(\frac{1}{2}\right) + \frac{v_0 t}{W} \frac{e^{2ip_0 L}}{2}$. By the time $t \geq T = \frac{W}{v_0}$, when the internal particle has been completely reflected and the wave packet ϕ_1 has moved away from the origin, $\langle \phi_1 | \alpha\delta(x_l) | \phi_2 \rangle = 0$ and the exchange of modular momentum ceases. We find that $\overline{e^{ip_h L}}(t \geq T) = \frac{1}{2}e^{i2p_0 L}$, namely, the modular momentum has changed. Thus the paradox is resolved thanks to the all important interference terms which were glossed over in introducing the paradox. The terms (11, 12) play here the role of the private potential difference between $x = 0$ and $x = L$. In analogy to (2), the private potential difference generates a nonlocal exchange of modular momentum. Indeed only a sufficiently strong interaction can change the state of the source by causing the internal light particle to be reflected. This conforms with our definition of a private potential as being experienced only by a strongly interacting particle. As we have seen, the private potential affects a finite change in the external particles's modular momentum or relative phase throughout the entanglement event.

Another Paradox seems to arise if we add to the set-up an infinite potential step at an intermediate point $0 < L_1 < L$. See fig. 4. The modular momentum of the external particle then changes in proportion to L_1 [14]. However, from the local point of view, if the external particle is in ψ_1 it totally reflects the internal particle, while if it is in ψ_2 , then its putative interaction with the internal particle is completely preempted by the potential step. In both cases, no information about the position L_1 of the barrier is conveyed to the external particle. How then can the external particle "know" the location of the barrier?

We shall now show that from the local perspective, focusing on the external heavy particle alone, the entanglement event is a generalized electric AB effect, which resolves the paradox. We have calculated the total change in modular momentum of the external particle during the scattering event by integrating (12) between $t = 0$ and $t = T$. Alternatively, the same change in the external particle's relative phase can be obtained as follows. Let the internal particle be initially in a very wide wave packet of width $W \gg L$ and of average momentum $p_{l,init} = p_0$ as above. However, unlike the previous set-up, where the external heavy particle was initially in a superposition of two wave packets located at $x = 0$ and $x = L$, respectively, here we assume that the external particle is in a single wave packet ψ centered around any point in the interval $0 \leq x \leq L$. The internal particle will then be reflected at x , with momentum $-p_0$, causing the momentum of the external particle to change by $+2p_0$. Recalling that $T = \frac{W}{v_0}$, the average momentum transferred after time t

$$\overline{\delta p_1}(x, t) = \bar{p}_1(x, t) - \bar{p}_1(x, 0) = \left(\frac{T-t}{T}\right) 0 + \left(\frac{t}{T}\right) 2p_0 = \int_0^t \bar{F}(x, t') dt' . \quad (13)$$

is the weighted average with probabilities $\frac{t}{T}$ and $\frac{T-t}{T}$ of the transfers corresponding to the reflections having occurred or not. The quantum average force which is constant both in x and in t , is therefore [15] $\bar{F} = \frac{2p_0}{T}$. We here use the average force to describe the underlying instantaneous but uncertain scattering events which involve singular forces. Since the external particle does not recoil, the only effect of a change of its momentum is a change of the gradient of its phase [16] or, $\bar{p}_1(x, t) = \frac{\partial \varphi}{\partial x}(x, t)$, $\bar{p}_1(x, 0) = \frac{\partial \varphi}{\partial x}(x, 0)$. Thus, for a continuous distribution of wave packets $\psi(x' - x)$ between $x = 0$ and $x = L$, pertaining to the external particle being located anywhere between $x = 0$ and $x = L$, we obtain for the time T when the scattering is over,

$$\varphi_{rel}(T) - \varphi_{rel}(0) = \int_0^T \int_0^L \bar{F} dx dt', \quad (14)$$

where $\varphi_{rel}(T) = \varphi(L, T) - \varphi(0, T)$ is then the relative phase between ψ_2 and ψ_1 . In our particular example $\varphi_{rel}(0) = 0$. Thus we have found an average force $\bar{F}(x) = \frac{2p_0}{T}$ which acts on the external particle when the latter is located at intermediate points x . This force when integrated according to (14) indeed yields the same relative phase as found in the previous section. This force which is, by definition, the gradient of a "private potential" provides the missing link and resolves our paradox. Specifically, the spatial part of the integral on the rhs of (14) ends at L_1 , the location of the barrier, beyond which the forces vanish. Thus the L_1 dependence is retrieved. The path integral on the rhs of (14) is analogous to the line integral $\int_0^L E dx$ of the electric field across the plates of a condenser responsible for the relative phase of the electric AB effect (2,3), with a crucial difference. Whereas the electric field between the plates of a condenser does not depend on the external electron in the electric AB effect, in our case, where the interaction with the external particle changes the state of the source, the force $\bar{F}(x, t)$ vanishes unless the external particle is actually located at x . Since the external particle is in ψ_1, ψ_2 , the rhs of (14) involves *potential* forces which, nevertheless, affect the change of the relative phase during the entanglement event. This is the central results of the present article, as illustrated by this simple scattering problem.

III. Summary and Conclusions

In the above we discussed a simple one dimensional scattering problem. The set-up consists of a heavy particle which is in a superposition of two narrow wave packets. The latter strongly interacts with a light incident particle which can be reflected from either of the two wave packets. This set-up leads to a novel, interesting, situation. It concerns the relative phase (between the two wave packets) of the heavy particle which is acquired during the entanglement /scattering event. This relative phase depends on virtual forces that would have acted on it had it been placed at intermediate points i.e; between

the two wave packets. This is so despite the fact that the heavy particle is restricted to the two stationary wave packets and never reaches the region in between. These forces, as elaborated in more detail in a following article in the context of the Born Oppenheimer approximation, derive from "private" potentials. Such potentials differ from the standard "public" potentials experienced by test particles. The reason is that the strongly interacting external heavy particle modifies the system it interacts with, which in turn modifies the interaction it experiences. Our analysis utilizes modular momentum and its exchange during the entanglement event.

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References

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- [4] However, the two wave packets can be positioned on the two respective sides of the (as yet closed) condenser only if the condenser plates are finite in at least one of the y and z directions. We therefore use a parallel-plate condenser with very large but finite plates as an approximation.
- [5] This feature is the defining property of "test particles" which are conventionally used in order to define the ordinary potentials. Such ordinary potentials are termed here "public potentials" in contradistinction to the "private potentials" introduced below.
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- [7] With $H \approx V(x)$, using $e^{ipL}e^{iF(x)} = e^{iF(x+L)}e^{ipL}$, we have
$$e^{ipL}(t) = e^{iHt}e^{ipL}(0)e^{-iHt} = e^{iV(x)t}e^{ipL}(0)e^{-iV(x)t} = e^{i[V(x)-V(x+L)]t}e^{ipL}(0). \quad (15)$$
- [8] The width of the tunnel d is small enough so that the lowest excitation in the transverse direction $\varepsilon_{transverse} = \frac{\hbar^2}{2md^2}$ is much larger than all other energy scales in the problem, and therefore the motion can be viewed as one dimensional.

- [9] With the Hamiltonian given by (6) and the state of the system by (7), multiplying the Schrödinger equation by $\langle \psi_1 |$ or $\langle \psi_2 |$, we obtain, respectively,

$$i \frac{\partial}{\partial t} | \phi_1 \rangle = \left[\frac{p_l^2}{2m} + \alpha \delta(x_l) \right] | \phi_1 \rangle \quad (16)$$

and

$$i \frac{\partial}{\partial t} | \phi_2 \rangle = \left[\frac{p_l^2}{2m} + \alpha \delta(x_l - L) \right] | \phi_2 \rangle \quad (17)$$

Jointly, equations (16) and (17) are equivalent to (8). These equations follow since the ψ 's do not depend on time, $\langle \psi_1 | \alpha \delta(x_l - x_h) | \psi_1 \rangle = \alpha \delta(x_l)$, $\langle \psi_2 | \alpha \delta(x_l - x_h) | \psi_2 \rangle = \alpha \delta(x_l - L)$, and $\langle \psi_i | \alpha \delta(x_l - x_h) | \psi_j \rangle = 0$ for $i \neq j$, since ψ_1, ψ_2 , are orthogonal at all times since there is no recoil.

- [10] *Reflection and transmission from $\alpha \delta(x)$, $\alpha \delta(x - L)$*
The energy eigenstates of the Hamiltonian $H = \frac{p^2}{2m} + \alpha \delta(x)$ for $x < 0$, $x > 0$, are, respectively,

$$\begin{aligned} \phi_L &= e^{ipx} + A e^{-ipx} \\ \phi_R &= B e^{ipx}. \end{aligned} \quad (18)$$

Boundary conditions:

a.

$$\phi_L(0) = \phi_R(0) \Rightarrow 1 + A = B. \quad (19)$$

b. Writing the stationary state Schrödinger equation

$$-\frac{\hbar^2}{2m} \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial x} \right) = [E - \alpha \delta(x)] \phi, \quad (20)$$

Integrating from $-\Delta$ to $+\Delta$, letting $\Delta \rightarrow 0$,

$$\left[\frac{\partial \phi_R}{\partial x} \right]_{x=0} - \left[\frac{\partial \phi_L}{\partial x} \right]_{x=0} = \tilde{\alpha} \phi(0), \quad (21)$$

where the dimensionless $\tilde{\alpha} \equiv \frac{2m\alpha}{\hbar^2}$. (18), (19) and (21) yield

$$\begin{aligned} A &= \frac{-1}{1 - \frac{2ip}{\tilde{\alpha}}} \rightarrow -1 \text{ as } \tilde{\alpha} \rightarrow \infty, \\ B &= 1 + A \approx -\frac{2ip}{\tilde{\alpha}} \rightarrow 0 \text{ as } \tilde{\alpha} \rightarrow \infty. \end{aligned} \quad (22)$$

With the δ function translated by L and ϕ_L, ϕ_R , now pertaining to $x < L, x > L$, respectively, A is multiplied by e^{2ipL} and B is unchanged.

- [11] At the origin, the internal particle is reflected from $\alpha\delta(x_l)$ and its wavefunction is $[10] \sim \frac{1}{\sqrt{W\alpha}}$. The average potential the heavy particle experiences there is then $\bar{V}(x_h = 0) \sim \alpha \left(\frac{1}{\sqrt{W\alpha}}\right)^2 = \frac{1}{W\alpha} \rightarrow 0$ as $\alpha \rightarrow \infty$. Similarly for $\bar{V}(x_h = L)$.

- [12] The complex conjugate of equation (16) is

$$-i\frac{\partial}{\partial t} \langle \phi_1 | = \langle \phi_1 | \left[\frac{p_l^2}{2m} + \alpha\delta(x_l) \right]. \quad (23)$$

- [13] The following is based on [10]. $\tilde{\alpha} \equiv \frac{2m\alpha}{\hbar^2}$, $\hbar \equiv 1$ thus $\tilde{\alpha} = 2m\alpha$. We have

$$\begin{aligned} \phi_1(x_l = 0) &\approx \frac{1}{\sqrt{W}}B = \frac{1}{\sqrt{W}}\left(-\frac{2ip_0}{\tilde{\alpha}}\right) = \frac{1}{\sqrt{W}}\left(-\frac{ip_0}{m\alpha}\right) \\ \phi_2(x_l = 0) &\approx \frac{1}{\sqrt{W}}\left(1 - e^{2ip_0L}\right), \end{aligned} \quad (24)$$

$$\begin{aligned} \frac{d}{dt}\overline{e^{ip_hL}} &= \frac{i}{2}\alpha\phi_1^*(x_l = 0)\phi_2(x_l = 0) = \frac{i}{2}\alpha\frac{1}{\sqrt{W}}\left(\frac{ip_0}{m\alpha}\right)\frac{1}{\sqrt{W}}\left(1 - e^{2ip_0L}\right) = -\frac{v_0}{W}\left(\frac{1}{2} - \frac{e^{2ip_0L}}{2}\right), \\ \overline{e^{ip_hL}}(t) &= \overline{e^{ip_hL}}(0) - \frac{v_0t}{W}\left(\frac{1}{2}\right) + \frac{v_0t}{W}\frac{e^{2ip_0L}}{2} = \frac{W - v_0t}{W}\left(\frac{1}{2}\right) + \frac{v_0t}{W}\frac{e^{2ip_0L}}{2}. \end{aligned} \quad (25)$$

- [14] $H_1 = H + V(x_l - L_1)$ where $V(x_l - L_1) = \infty$, $x_l - L_1 \geq 0$, and $= 0$ otherwise. With the external particle in ψ_1 , the internal particle sees the potential $\alpha\delta(x) + V(x - L_1)$. At the limit $\alpha \rightarrow \infty$ the energy eigenstate for $x < 0$ is

$$\phi_1 = \sin px. \quad (27)$$

With the xp in ψ_2 , the internal particle sees the infinite potential step alone. The solution for $x \leq L_1$ is

$$\phi_2 = \sin p(x - L_1), \quad (28)$$

Thus

$$\Psi(t) = c_1(t)(\psi_1 + \psi_2)e^{ipx} + c_2(t)(\psi_1 + e^{2ipL_1}\psi_2)e^{-ipx}. \quad (29)$$

- [15] Let $A = \sum_{n=0}^N A_n$. For Ψ an eigenstate of A with eigenvalue a we have $\bar{A}a = a = \sum_{n=0}^N \bar{A}_n a$. And similarly for $I = \int_0^T F dt$. Over an eigenstate of I , we have $\bar{I} = 2p_0 = \overline{\int_0^T F dt} = \int_0^T \bar{F} dt$. Assuming a constant force, we obtain $\bar{F}T = 2p_0$.

[16] The heavy, i.e; not recoiling, external particle may be described by a stationary Gaussian wavepacket. With the external particle located at $0 \leq x_0 \leq L$ we have

$$\psi(x - x_0) = n e^{-\frac{(x-x_0)^2}{2\Delta^2}} e^{i\varphi(x)} \equiv n e^{-f(x-x_0)} e^{i\varphi(x)}, \quad (30)$$

such that $\int \psi^* \psi dx = 1$. $\bar{p}_1 = \int \psi^* \frac{1}{i} \frac{\partial \psi}{\partial x} dx$. The imaginary part vanishes. The real part equals $\bar{p}_1 = \frac{\partial \varphi}{\partial x}(x_0)$ since the very narrow normalized Gaussian distribution $n^2 e^{-2f(x-x_0)}$ behaves like a δ function at x_0 .

Captions for Figures

Fig. 1. Set-up for the electric AB Effect.

Fig. 2. The internal particle, moving inside the tunnel, is reflected at either $x=0$ or at $x=L$.

Fig. 3. The external particle is in a superposition of two narrow, stationary wave packets ψ_1, ψ_2 . The internal particle is initially in a very wide wave packet ϕ_{in} , moving with velocity v_0 .

Fig. 4. Set-up for 2nd paradox, where an additional infinitely heavy barrier has been introduced at L_1 .